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On Jordan Triple Homomorphism and Generalized Jordan Triple Homomorphism of Gamma Rings

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Abstract:

Let M and M' be two Γ-rings , in the present paper we introduced the concepts of Jordan triple homomorphism , generalized Jordan triple homomorphism on Γ-rings and some Lemmas .

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Key Words: Γ– ring , Jordan homomorphism , Jordan triple homomorphism .

1- Introduction:

Let M and Γ be two additive abelian groups, suppose that there is a mapping from $M\times\Gamma\times M \longrightarrow M$ (the image of (a,α,b) being denoted by $a\alpha b$, $a, b \in M$ and $\alpha \in \Gamma$). Satisfying for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$

$a(\alpha + \beta)c = a\alpha c + a\beta c$

$a\alpha(b + c) = a\alpha b + a\alpha c$

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$

Then M is called a Γ-ring. This definition is due to Barnes [1].

Let M be a Γ-ring, then M is called 2-torsion free if $2a = 0$ implies that $a = 0$, for all $a \in M$. This definition is due to [2].

An additive mapping θ of a Γ-ring M into a Γ-ring M' is called homomorphism if $\theta(a\alpha b) = \theta(a)\alpha\theta(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$. This definition is due to [1].

An additive mapping θ of Γ-ring M into a Γ-ring M' is called Jordan homomorphism if $\theta(a\alpha b + b\alpha a) = \theta(a)\alpha\theta(b) + \theta(b)\alpha\theta(a)$, for all $a, b \in M$ and $\alpha \in \Gamma$. This definition is due to [3].

Let F be an additive mapping of a Γ-ring M into a Γ-ring M'. F is called a generalized homomorphism if there exists a homomorphism θ from a Γ-ring M into a Γ-ring M', such that $F(a\alpha b) = F(a)\alpha\theta(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$, where θ is called the relating homomorphism . This definition is due to [3].

And F is called a generalized Jordan homomorphism if there exists a Jordan homomorphism θ from a Γ-ring M into a Γ-ring M', such that

$F(a\alpha b + b\alpha a) = F(a)\alpha\theta(b) + F(b)\alpha\,\theta(a)$, for all $a, b \in$ M and $\alpha \in \Gamma$, where θ is called the relating Jordan homomorphism. This definition is due to [3].

2- Jordan Triple Homomorphism of Γ-Rings :

Definition (2.1):

An additive mapping θ of Γ-ring M into a Γ-ring M' is called Jordan triple homomorphism if $\theta(a\alpha b\beta a) = \theta(a)\alpha\theta(b)\beta\theta(a)$, for all $a, b \in$ M and $\alpha,\beta \in \Gamma$.

Example (2.2):

Let R be a ring , let M = $M_{1\times2}(R)$, M' = $M'_{1\times2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix}, n \in Z \right\}$ then M and M' be Γ-rings. Let θ be an additive mapping of a Γ-ring M into a Γ-ring M', such that $\theta((a \quad b)) = (a \quad 0)$, for all $(a \quad b) \in$ M we obtain θ is a Jordan triple homomorphism

Lemma (2.3):

Let θ be a Jordan triple homomorphism of a Γ-ring M into a Γ- ring M ', then for all $a, b, c \in$ M, $\alpha, \beta \in \Gamma$ and $n \in N$

(i) $\theta(a\alpha b\beta a + a\beta b\alpha a) = \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\beta\theta(b)\alpha\theta(a)$

(ii) $\theta(a\alpha b\beta c + c\alpha b\beta a) = \theta(a)\alpha\theta(b)\beta\theta(c) + \theta(c)\alpha\theta(b)\beta\theta(a)$

(iii) In particular, if M, M' be two commutative Γ-rings and M' is a 2-torsion free Γ-ring , then

$\theta(a\alpha b\beta c) = \theta(a)\alpha\theta(b)\beta\theta(c)$

(iv) $\theta(a\alpha b\alpha c + c\alpha b\alpha a) = \theta(a)\alpha\theta(b)\alpha\theta(c) + \theta(c)\alpha\theta(b)\alpha\theta(a)$

Proof:

(i) Replace $a\beta b + b\beta a$ for b in Definition Jordan homomorphism , we get :

$\theta(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \theta(a)\alpha\theta(a\beta b + b\beta a) + \theta(a\beta b + b\beta a)\alpha\theta(a)$

$= \theta(a)\alpha\theta(a)\beta\theta(b) + \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\beta\theta(b)\alpha\theta(a) + \theta(b)\beta\theta(a)\alpha\theta(a)$ …(1)

On the other hand

$\theta(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \theta(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$

$= \theta(a)\alpha\theta(a)\beta\theta(b) + \theta(b)\beta\theta(a)\alpha\theta(a) + \theta(a\alpha b\beta a + a\beta b\alpha a)$ …(2)

Compare (1) and (2), we get:

$\theta(a\alpha b\beta a + a\beta b\alpha a) = \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\beta\theta(b)\alpha\theta(a)$

(ii) Replace $a + c$ for a in Definition (2.1), we get:

$\theta((a + c)\alpha b\beta(a + c)) = \theta(a + c)\alpha\theta(b)\beta\theta(a + c)$

$= \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\alpha\theta(b)\beta\theta(c) + \theta(c)\alpha\theta(b)\beta\theta(a) + \theta(c)\alpha\theta(b)\beta\theta(c)$ … (1)

On the other hand

$\theta((a + c)\alpha b\beta(a + c)) = \theta(a\alpha b\beta a + a\alpha b\beta c + c\alpha b\beta a + c\alpha b\beta c)$

$=\theta(a)\alpha\theta(b)\beta\theta(a)+\theta(c)\alpha\theta(b)\beta\theta(c)+\theta(a\alpha b\beta c+c\alpha b\beta a)$ …(2)

Compare (1) and (2), we get:

$\theta(a\alpha b\beta c+c\alpha b\beta a)=\theta(a)\alpha\theta(b)\beta\theta(c) + \theta(c)\alpha\theta(b)\beta\theta(a)$

(iii) By (ii) and since M, M' be two commutative Γ-rings and M' is a 2-torsion free Γ-ring , then

$\theta(a\alpha b\beta c + a\alpha b\beta c) = 2\theta(a\alpha b\beta c) = \theta(a)\alpha\theta(b)\beta\theta(c)$

(iv) Replace α for β in (ii), we get:

$\theta(a\alpha b\alpha c+c\alpha b\alpha a)=\theta(a)\alpha\theta(b)\alpha\theta(c)+ \theta(c)\alpha\theta(b)\alpha\theta(a)$

Definition (2.4):

Let θ be a Jordan homomorphism of a Γ-ring M into a Γ-ring M',then for all $a, b \in$ M and $\alpha \in \Gamma$, we define

$G(a,b,a)_{\alpha,\beta} = \theta(a\alpha b\beta a) - \theta(a)\alpha\theta(b)\beta\theta(a)$.

Lemma (2.5):

If θ be a Jordan triple homomorphism of a Γ-ring M into a Γ-ring M',then for all a, b,c,d $\in$ M and $\alpha, \beta \in \Gamma$

(i) $G((a + b),c,d)_{\alpha,\beta} = G(a,c,d)_{\alpha,\beta} + G(b,c,d)_{\alpha,\beta}$

(ii) $G(a,(b + c),d)_{\alpha,\beta} = G(a,b,d)_{\alpha,\beta} + G(a,c,d)_{\alpha,\beta}$

(iii) $G(a,b,(c+ d))_{\alpha,\beta} = G(a,b,c)_{\alpha,\beta} + G(a,b,d)_{\alpha,\beta}$

Proof:

(i) $G((a + b),c,d)_{\alpha,\beta} = \theta((a + b)\alpha c\beta d) - \theta(a + b)\alpha\theta(c)\beta\theta(d)$

$= \theta(a\alpha c\beta d) - \theta(a)\alpha\theta(c)\beta\theta(d) + \theta(b\alpha c\beta d) - \theta(b)\alpha\theta(c)\beta\theta(d)$

$= G(a,c,d)_{\alpha,\beta} + G(b,c,d)_{\alpha,\beta}$

(ii) $G(a,(b + c),d)_{\alpha,\beta} = \theta(a\alpha(b + c)\beta d) - \theta(a)\alpha\theta(b + c)\beta\theta(d)$

$= \theta(a\alpha b\beta d) - \theta(a)\alpha\theta(b)\beta\theta(d) + \theta(a\alpha c\beta d) - \theta(a)\alpha\theta(c)\beta\theta(d)$

$= G(a,b,d)_{\alpha,\beta} + G(a,c,d)_{\alpha,\beta}$

(iii) $G(a,b,(c+ d))_{\alpha,\beta} = \theta(a\alpha b\beta(c + d)) - \theta(a)\alpha\theta(b)\beta\theta(c + d)$

$= \theta(a\alpha b\beta c) - \theta(a)\alpha\theta(b)\beta\theta(c) + \theta(a\alpha b\beta d) - \theta(a)\alpha\theta(b)\beta\theta(d)$

$= G(a,b,c)_{\alpha,\beta} + G(a,b,d)_{\alpha,\beta}$

Proposition (2.6):

Let θ be a Jordan homomorphism from a Γ-ring M into a 2-torsion free Γ-ring M', such that $a\alpha b\beta a = a\beta b\alpha a$, for all $a, b \in$ M and $\alpha, \beta \in \Gamma$, $a'\alpha b'\beta a' = a'\beta b'\alpha a'$, for all $a', b' \in$ M' and $\alpha, \beta \in \Gamma$.Then θ is Jordan triple homomorphism.

Proof:

Replace b by $a\beta b + b\beta a$ in Definition of Jordan homomorphism , we get :

$\theta(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \theta(a)\alpha\theta(a\beta b + b\beta a) + \theta(a\beta b + b\beta a)\alpha\theta(a)$

$= \theta(a)\alpha\theta(a)\beta\theta(b) + \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\beta\theta(b)\alpha\theta(a) + \theta(b)\beta\theta(a)\alpha\theta(a)$

Since $a'\alpha b'\beta a' = a'\beta b'\alpha a'$, for all $a', b' \in$ M' and $\alpha, \beta \in \Gamma$, we get:

$= \theta(a)\alpha\theta(a)\beta\theta(b)+2\theta(a)\alpha\theta(b)\beta\theta(a)+\theta(b)\beta\theta(a)\alpha\theta(a)$ …(1)

On the other hand:

$\theta(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \theta(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$

Since $a\alpha b\beta a = a\beta b\alpha a$, for all $a, b \in$ M and $\alpha, \beta \in \Gamma$, we get:

$= \theta(a\alpha a\beta b + b\beta a\alpha a) + 2\theta(a\alpha b\beta a)$

$= \theta(a)\alpha\theta(a)\beta\theta(b) + \theta(b)\beta\theta(a)\alpha\theta(a) + 2\theta(a\alpha b\beta a)$ …(2)

Compare (1) and (2), we get:

$2\theta(a\alpha b\beta a) = 2\theta(a)\alpha\theta(b)\beta\theta(a)$.

Since M' is 2-torsion free Γ-ring , we obtain that θ is Jordan triple homomorphism.

3- Generalized Jordan Triple Homomorphism of Γ-Rings :

Definition (3.1):

An additive mapping F of a Γ-ring M into a Γ-ring M' is called a generalized Jordan triple homomorphism if there exists a Jordan triple homomorphism θ from a Γ-ring M into a Γ-ring M' such that

$F(a\alpha b\beta a) = F(a)\alpha\theta(b)\beta\theta(a)$, for all $a, b \in$ M and $\alpha , \beta \in \Gamma$.

Where θ is called the relating Jordan triple homomorphism.

Example (3.2):

Let R be a ring, let M = $M_{1\times 2}(R)$, M' = $M_{1\times 2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix}, n \in Z \right\}$.Then M and M' be Γ-rings.Let F be an additive mapping of a Γ-ring M into a Γ-ring M', such that $F((a \quad b)) = (-a \quad 0)$, for all $(a \quad b) \in M$,then there exists a homomorphism θ from a Γ-ring M into a Γ-ring M', such that $\theta((a \quad b)) = (a \quad 0)$, for all $(a \quad b) \in M$. Then F is a generalized Jordan triple homomorphism

Lemma (3.3):

Let θ be a generalized Jordan triple homomorphism of a Γ-ring M into a Γ- ring M ', then for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$ and $n \in N$

(i) $F(a\alpha b\beta a + a\beta b\alpha a)=F(a)\alpha\theta(b)\beta\theta(a) + F(a)\beta\theta(b)\alpha\theta(a)$

(ii) $F(a\alpha b\beta c+c\alpha b\beta a)= F(a)\alpha\theta(b)\beta\theta(c) + F(c)\alpha\theta(b)\beta\theta(a)$

(iii) In particular, if M, M' be two commutative Γ-rings and M' is a 2-torsion free Γ-ring , then

$F(a\alpha b\beta c) = F(a)\alpha\theta(b)\beta\theta(c)$

(iv) $F(a\alpha b\alpha c +c\alpha b\alpha a)=F(a)\alpha\theta(b)\alpha\theta(c)+ F(c)\alpha\theta(b)\alpha\theta(a)$

Proof:

(i) Replace $a\beta b + b\beta a$ for b in Definition generalized Jordan homomorphism , we get:

$F(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = F(a)\alpha\theta(a\beta b + b\beta a) + F(a\beta b + b\beta a)\alpha\theta(a)$

$= F(a)\alpha\theta(a)\beta\theta(b)) + F(a)\alpha\theta(b)\beta\theta(a)) + F(a)\beta\theta(b)\alpha\theta(a) + F(b)\beta\theta(a)\alpha\theta(a)$

$= F(a)\alpha\theta(a)\beta\theta(b) + F(a)\alpha\theta(b)\beta\theta(a) + F(a)\beta\theta(b)\alpha\theta(a) + F(b)\beta\theta(a)\alpha\theta(a) \quad \ldots(1)$

On the other hand

$F(a\alpha(a\beta b + b\beta a)+(a\beta b + b\beta a)\alpha a)=F(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$

$= F(a)\alpha\theta(a)\beta\theta(b)+F(b)\beta\theta(a)\alpha\theta(a) + F(a\alpha b\beta a+a\beta b\alpha a) \quad \ldots(2)$

Compare (1) and (2), we get:

$F(a\alpha b\beta a+ a\beta b\alpha a)=F(a)\alpha\theta(b)\beta\theta(a) + F(a)\beta\theta(b)\alpha\theta(a)$

(ii) Replace $a + c$ for a in Definition (3.1) , we get:

$F((a + c)\alpha b\beta(a + c)) = F(a + c)\alpha\theta(b)\beta\theta(a + c)$

$= F(a)\alpha\theta(b)\beta\theta(a) + F(a)\alpha\theta(b)\beta\theta(c) + F(c)\alpha\theta(b)\beta\theta(a) + F(c)\alpha\theta(b)\beta\theta(c) \ldots(1)$

On the other hand

$F((a + c)\alpha b\beta(a + c)) = F(a\alpha b\beta a + a\alpha b\beta c + c\alpha b\beta a + c\alpha b\beta c)$

$=F(a)\alpha\theta(b)\beta\theta(a)+F(c)\alpha\theta(b)\beta\theta(c)+F(a\alpha b\beta c+ c\alpha b\beta a)$ …(2)

Compare (1) and (2), we get:

$F(a\alpha b\beta c+c\alpha b\beta a) = F(a)\alpha\theta(b)\beta\theta(c) + F(c)\alpha\theta(b)\beta\theta(a)$

(iii) By (ii) and since M, M' be two commutative Γ-rings and M' is a 2-torsion free Γ-ring

$F(a\alpha b\beta c + a\alpha b\beta c) = 2F(a\alpha b\beta c) = F(a)\alpha\theta(b)\beta\theta(c)$

(iv) Replace α for β in (ii), we get:

$F(a\alpha b\alpha c + c\alpha b\alpha a) = F(a)\alpha\theta(b)\alpha\theta(c) + F(c)\alpha\theta(b)\alpha\theta(a)$

<u>Definition (3.4):</u>

Let F be a generalized Jordan homomorphism of a Γ-ring M into a Γ-ring M',then for all $a, b \in$ M and $\alpha \in \Gamma$, we define

$\delta\,(a,b,a)_{\alpha,\beta} = F(a\alpha b\beta a) – F(a)\alpha\theta(b)\beta\theta(a)$.

<u>Lemma (3.5):</u>

If F be a generalized Jordan triple homomorphism of a Γ-ring M into a Γ-ring M',then for all $a, b,c,d \in$ M and $\alpha, \beta \in \Gamma$

(i) $\delta\,((a + b),c,d)_{\alpha,\beta} = \delta\,(a,c,d)_{\alpha,\beta} +\delta\,(b,c,d)_{\alpha,\beta}$

(ii) $\delta\,(a,(b + c),d)_{\alpha,\beta} = \delta\,(a,b,d)_{\alpha,\beta} + \delta\,(a,c,d)_{\alpha,\beta}$

(iii) $\delta\,(a,b,(c+ d))_{\alpha,\beta} = \delta\,(a,b,c)_{\alpha,\beta} + \delta\,(a,b,d)_{\alpha,\beta}$

<u>Proof:</u>

(i) $\delta\,((a + b),c,d)_{\alpha,\beta} = F((a + b)\alpha c\beta d) – F(a + b)\alpha\theta(c)\beta\theta(d)$

$= F(a\alpha c\beta d) – F(a)\alpha\theta(c)\beta\theta(d) + F(b\alpha c\beta d) – F(b)\alpha\theta(c)\beta\theta(d)$

$= \delta\,(a,c,d)_{\alpha,\beta} + \delta\,(b,c,d)_{\alpha,\beta}$

(ii) $\delta\,(a,(b + c),d)_{\alpha,\beta} = F(a\alpha(b + c)\beta d) – F(a)\alpha\theta(b + c)\beta\theta(d)$

$= F(a\alpha b\beta d) – F(a)\alpha\theta(b)\beta\theta(d) + F(a\alpha c\beta d) – F(a)\alpha\theta(c)\beta\theta(d)$

$= \delta\,(a,b,d)_{\alpha,\beta} + \delta\,(a,c,d)_{\alpha,\beta}$

(iii) $\delta\,(a,b,(c+ d))_{\alpha,\beta} = F(a\alpha b\beta(c + d)) – F(a)\alpha\theta(b)\beta\theta(c + d)$

$= F(a\alpha b\beta c) – F(a)\alpha\theta(b)\beta\theta(c) + F(a\alpha b\beta d) – F(a)\alpha\theta(b)\beta\theta(d)$

$= \delta\,(a,b,c)_{\alpha,\beta} + \delta\,(a,b,d)_{\alpha,\beta}$

<u>Proposition (3.6):</u>

Let F be a generalized Jordan homomorphism from a Γ-ring M into a 2-torsion free Γ-ring M', such that $a\alpha b\beta a = a\beta b\alpha a$, for all $a, b \in$ M and $\alpha, \beta \in \Gamma$,

$a'\alpha b'\beta a' = a'\beta b'\alpha a'$, for all $a', b' \in M'$ and $\alpha, \beta \in \Gamma$.Then F is a generalized Jordan triple homomorphism.

Proof:

Replace b by $a\beta b + b\beta a$ in Definition of generalized Jordan homomorphism , we get:

$F(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = F(a)\alpha\theta(a\beta b + b\beta a) + F(a\beta b + b\beta a)\alpha\theta(a)$

$= F(a)\alpha\theta(a)\beta\theta(b) + F(a)\alpha\theta(b)\beta\theta(a) + F(a)\beta\theta(b)\alpha\theta(a) + F(b)\beta\theta(a)\alpha\theta(a)$

Since $a'\alpha b'\beta a' = a'\beta b'\alpha a'$, for all $a', b' \in M'$ and $\alpha, \beta \in \Gamma$, we get:

$=F(a)\alpha\theta(a)\beta\theta(b) + 2F(a)\alpha\theta(b)\beta\theta(a) + F(b)\beta\theta(a)\alpha\theta(a)$ …(1)

On the other hand:

$F(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = F(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$

Since $a\alpha b\beta a = a\beta b\alpha a$, for all $a, b \in M$ and $\alpha, \beta \in \Gamma$, we get:

$= F(a)\alpha\theta(a)\beta\theta(b) + F(b)\beta\theta(a)\alpha\theta(a) + 2F(a\alpha b\beta a)$ …(2)

Compare (1) and (2), we get:

$2F(a\alpha b\beta a) = 2F(a)\alpha\theta(b)\beta\theta(a)$.

Since M' is 2-torsion free Γ-ring , we get F is a generalized Jordan triple homomorphism.

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